Nov. 8. Question set.

Question 1.

**Question**

What is the elasticity of scale of the CES technology, $f(x_1, x_2) = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}}$ ?

It is easy to show the elasticity of scale of a CES production function is 1.

By defination

$$\varepsilon = \lim_{t \to 1} \frac{d \ln[f(tx)]}{d \ln t} = \frac{\sum f_i(x) \cdot x_i}{f(x)}$$

$$= \frac{\frac{1}{\rho}(x_1^\rho + x_2^\rho)^{\frac{1}{\rho}-1} \cdot \rho x_1^\rho + \frac{1}{\rho}(x_1^\rho + x_2^\rho)^{\frac{1}{\rho}-1} \rho \cdot x_2^\rho}{(x_1^\rho + x_2^\rho)^{\frac{1}{\rho}}}$$

$$= \frac{(x_1^\rho + x_2^\rho)^{\frac{1}{\rho}}}{(x_1^\rho + x_2^\rho)^{\frac{1}{\rho}}} = 1.$$

Question: 2020 Prelim OCT. Part 1. Q3.

**Question** 2020 Oct Prelim Part1 Q3

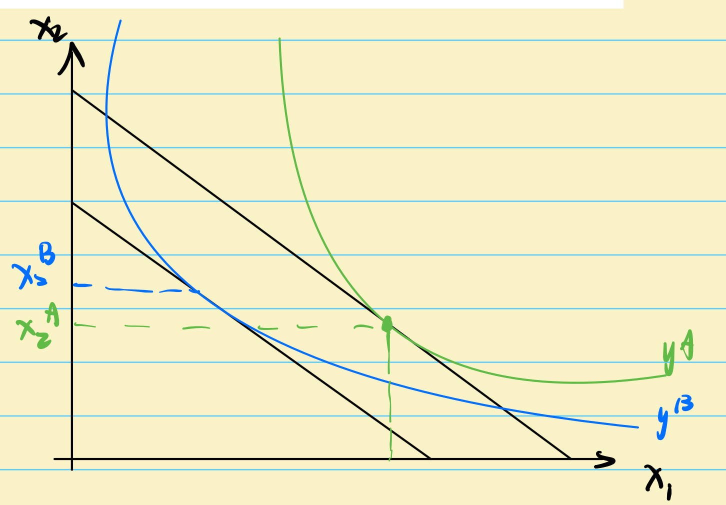
A factor of production $i$ is called inferior if the conditional demand for that factor decreases as output increases; that is, $\partial x_i(\mathbf{w}, y)/\partial y < 0$.

(a) Draw a diagram indicating that inferior factors are possible.

**(b) Show that if the technology is constant returns to scale, then no factors can be inferior.**

(c) Show that if marginal cost decreases as the price of some factor increases, then that factor must be inferior.

(a)




$$\frac{\partial x_2(w, y)}{\partial y} < 0.$$

$x_2$ is an inferier input.

JR 3.31.

**Question JR 3.31**

The output elasticity of demand for input $x_i$ is defined as

$$\epsilon_{iy}(w, y) \equiv (\partial x_i(\mathbf{w}, y)/\partial y)\,(y/x_i(\mathbf{w}, y)). \qquad (3)$$

(a) Show that $\epsilon_{iy}(\mathbf{w}, y) = \phi(y)\epsilon_{iy}(\mathbf{w}, 1)$ when the production function is homothetic.
(b) Show that $\epsilon_{iy} = 1$, for $i = 1, \ldots, n$, when the production function has constant returns to scale.

(a) When the production is homothetic.

According to JR Theorem 3.4, $x_i(w,y) = h(y)\, x_i(w,1)$

Therefore, $\varepsilon_{iy}(w,y) = \frac{\partial x_i}{\partial y} \cdot \frac{y}{x_i} = h'(y)\, x_i(w,1) \frac{y}{x_i(w,y)}$

$= h'(y)\, \cancel{x_i(w,1)} \frac{y}{h(y)\, \cancel{x_i(w,1)}}$

$= \frac{h'(y) \cdot y}{h(y)}$

Then, $\varepsilon_{iy}(w,1) = \frac{h'(1)}{h(1)}$

So, we define $\varphi(y) = \frac{h(1)}{h'(1)} \cdot \frac{h'(y)}{h(y)} \cdot y$

$\Rightarrow \varepsilon_{iy}(w,y) = \frac{h'(y)\, y}{h(y)} = \frac{h(1)}{h'(1)} \cdot \frac{h'(y)}{h(y)} \cdot y \cdot \frac{h'(1)}{h(1)} = \phi(y) \cdot \varepsilon_{iy}(w,1)$

(b) When production has CRS $\Rightarrow f(tx) = t^{\alpha} f(x)$, $\alpha = 1.$, $h(y) = y$

$\varepsilon_{iy} = \frac{h'(y)\, y}{h(y)} = \frac{y}{y} = 1$

(b) Show that if technology is constant return to scale, then no factors can be inferior.

CRS : $f(tx) = t f(x)$ $\Rightarrow$ Homogeneous of degree 1.

$\Rightarrow$ homogeneous production function

$\Rightarrow$ Also homothetic.

Therefore, applying JR Theorem 3.4.

the conditional input: $x_i(w, y) = y \cdot x_i(w, 1)$

Hence, $\frac{\partial x_i(w, y)}{\partial y} = x_i(w, 1) \geq 0, \forall i$ no inferior inputs?

(c) $\frac{\partial MC}{\partial w_i} = \frac{\partial^2 c}{\partial y \partial w_i} = \underbrace{\frac{\partial\left(\frac{\partial c}{\partial w_i}\right)}{\partial y} = \frac{\partial x_i}{\partial y}}_{\text{using the Hotelling's lemma.}} < 0 \Leftrightarrow$ input $i$ is inferior

**Question - JR 3.28**

A firm's technology possesses all the usual properties. It produces output using three inputs, with conditional input demands $x_i(w_1, w_2, w_3, y), i = 1, 2, 3$. Some of the following observations are consistent with cost minimization and some are not. If an observation is inconsistent, explain why. If it is consistent, give an example of a cost or production function that would produce such behavior.

(a) $\partial x_2/\partial w_1 > 0$ and $\partial x_3/\partial w_1 > 0$.
(b) $\partial x_2/\partial w_1 > 0$ and $\partial x_3/\partial w_1 < 0$.
(c) $\partial x_1/\partial y < 0$ and $\partial x_2/\partial y < 0$ and $\partial x_3/\partial y < 0$.
(d) $\partial x_1/\partial y = 0$.
(e) $\partial (x_1/x_2)/\partial w_3 = 0$

(a) Consistent

When production function is C-D. i.e. $y = x_1^{\alpha} x_2^{\beta} x_3^{1-\alpha-\beta}$

the cost function is $c(w_1, w_2, w_3, y) = y\left(\frac{w_1}{\alpha}\right)^{\alpha}\left(\frac{w_2}{\beta}\right)^{\beta}\left(\frac{w_3}{\rho}\right)^{\rho}$

(let $\rho = 1-\beta-\alpha$)

therefore, $\frac{\partial x_2}{\partial w_1} = \frac{\partial^2 c}{\partial w_2 \partial w_1} > 0$ and $\frac{\partial x_3}{\partial w_1} = \frac{\partial^2 c}{\partial w_3 \partial w_1} > 0$

(b) consistent

for example, $f(x_1, x_2, x_3) = x_2 \min\{x_1, x_3\}$

In this case, the conditional input demand are

$$x_2 = \sqrt{y\frac{w_1+w_3}{w_2}}, \quad x_1 = x_3 = \sqrt{y\frac{w_2}{w_1+w_3}}$$

then $\frac{\partial x_2}{\partial w_1} > 0$ $\frac{\partial x_3}{\partial w_1} < 0$

(c) $\frac{\partial x_1}{\partial y} < 0$ $\frac{\partial x_2}{\partial y} < 0$ $\frac{\partial x_3}{\partial y} < 0$ are not consistent with cost min.

Recall $c(w_1, w_2, w_3, y) = w_1 x_1(w, y) + w_2 x_2(w, y) + w_3 x_3(w, y)$

$$\Rightarrow \frac{\partial c}{\partial y} = w_1\frac{\partial x_1}{\partial y} + w_2\frac{\partial x_2}{\partial y} + w_3\frac{\partial x_3}{\partial y}$$

$\frac{\partial x_i}{\partial y} < 0$ for $\forall i \in \{1, 2, 3\}$ $\Rightarrow \frac{\partial c}{\partial y} < 0$

which is contradictory to the fact that cost is increasing in $y$.

(d) $\frac{\partial x_1}{\partial y} = 0$ is not consistent with cost minimization as well.

If $\frac{\partial x_1}{\partial y} = 0 \Rightarrow x_1$ will not change when $y$ changes.

As $f(0) = 0 \Rightarrow$ when $y = 0 \Rightarrow x = 0$

Therefore. $x_1$ must be always 0. at each possible value of $y$.

$\Rightarrow$ contradictory to the fact that the firm has three inputs.

( we don't need $x_1$ as input in this case )

(e) $\frac{\partial (\frac{x_1}{x_2})}{\partial w_3} = 0$ this is consistent with cost minimization.

( something similar to the independence of irrelevant assumption)

Consider a Cobb-Douglas production function

$$c(w, y) = y \cdot \left(\frac{w_1}{\alpha}\right)^{\alpha} \left(\frac{w_2}{\beta}\right)^{\beta} \left(\frac{w_3}{v}\right)^{v}$$

then $$x_1 = y \left(\frac{w_1}{\alpha}\right)^{\alpha - 1} \left(\frac{w_2}{\beta}\right)^{\beta} \left(\frac{w_3}{v}\right)^{v}$$

$$x_2 = y \left(\frac{w_1}{\alpha}\right)^{\alpha} \left(\frac{w_2}{\beta}\right)^{\beta - 1} \left(\frac{w_3}{v}\right)^{v}$$

$\frac{x_1}{x_2}$ is irrelevent to $w_3$! $\Leftrightarrow \frac{\partial (\frac{x_1}{x_2})}{\partial w_3} = 0$

JR 4.23 Ramsey Rule.

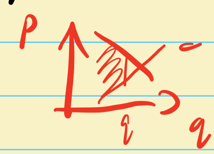
Q: Suppose a monopolist face negatively slope demand $p = p(q)$ and has cost $C = cq + F$

Now. Suppose the gov requires this firm to set a price $(p^*)$ that max the sum of consumer & producer's surplus. s.t. firm's revenue is non-negative

Show ① firm will charge a price higher than marginal cost.
② and $\frac{p^* - c}{p^*}$ will be proportionate to $\frac{1}{\varepsilon^*}$.

① The sum of consumer & producer welfare is

$$\int_0^q (p(t) - c)\, dt - F$$




then, firm optimize:

$$\max_q \int_0^q (p(t) - c)\, dt - F$$

$$\text{s.t. } p(q)\cdot q - cq - F \geq 0$$ ← this is an unconstraint optimization.

$$\mathcal{L} = \int_0^q [p(t) - c]\cdot dt - F + \lambda[p(q)\cdot q - cq - F]$$

KKT conditions:

$$[q]: \; p(q) - c \quad + \lambda[p'(q)\cdot q + p(q) - c] = 0$$

$$\lambda[p(q)\cdot q - cq - F] = 0$$

$$\lambda \geq 0 \qquad p(q)\cdot q - cq - F \geq 0$$

Now, discuss by case.

If $\lambda = 0 \Rightarrow p^*(q) = c \Rightarrow \pi = p^*(q)\cdot q^* - cq^* - F = -F < 0$
violates the requirement of non-negative profit!

Hence $\lambda > 0$, $p(q) \cdot q - c \cdot q - F = 0$

$$\Rightarrow p^* = \frac{cq + F}{q} = c + \frac{F}{q} > c$$

The optimal price $p^*$ will be strictly larger than the MC

Reason: b.c. there is a fixed cost F. if firm set $p = MC$
then the profit will be negative.
To obtain a non-negative profit, firms need to set a higher price than marginal cost.

② As $(p^*_{(q^*)} - c)\, q^* = F$
take partial derivitive w.r.t. $q^*$

$$p^{*\prime}(q^*) \cdot q^* + p^*(q^*) - c = 0$$

$$\Rightarrow \frac{p^* - c}{p} = -\frac{p^{*\prime} \cdot q^*}{p^*} = -\frac{1}{\varepsilon}$$

This implies that with ~~less elastic~~ more inelastic consumer demand, the percentage deviation $\frac{p^* - c}{p^*}$ will be higher

$\Rightarrow$ firm has larger mkt power!